Final examination 2019
M.Math. II — Algebraic Geometry
Each question carries 20 marks.

Group A

Answer any one question from group A

Question 1
Let V be a non-empty variety in $\mathbb{A}^n$, $\Gamma(V)$ be the coordinate ring. For $P \in V$ let $\mathcal{O}_P(V)$ be the set of rational functions on V that are defined at P.
(a) Show that $\Gamma(V) = \cap_{P \in V} \mathcal{O}_P(V)$.
(b) Show that $\mathcal{O}_P(V)$ is a noetherian local domain.

Question 2
Let $\varphi : V \longrightarrow W$ be a polynomial map of affine varieties and $\tilde{\varphi} : \Gamma(W) \longrightarrow \Gamma(V)$ be the induced map of coordinate rings. Suppose $P \in V, \varphi(P) = Q$.
(a) Show that $\tilde{\varphi}$ extends uniquely to a ring homomorphism from $\mathcal{O}_Q(W)$ to $\mathcal{O}_P(V)$.
(b) Show that $\tilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$.

Group B

Answer any four questions from group B

Question 3
(a) Let P be a simple point on a plane curve F in the projective plane. Show that the tangent line to F at P has the equation $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$.
(b) Show that the curve $X^2Y^3 + X^2Z^3 + Y^2Z^3$ is irreducible. Find its multiple points and the multiplicities and tangent lines at the multiple points.

Question 4
(a) Let $P \in \mathbb{P}^2$. Let F, G be two curves with no common component through P and H be another curve. State Noether's conditions with respect to the curves F, G, H at P.
(b) Let F, G, H be plane curves and $P \in F \cap G$. Show that Noether's conditions are satisfied at P if either of the following is true:
(i) P is a simple point on F and $I(P, H \cap F) \geq I(P, G \cap F)$.
(ii) F and G have distinct tangents at P and $m_P(H) \geq m_P(F) + m_P(G) - 1$.

Question 5
Let V be an affine variety and let $f \in \Gamma(V), f \neq 0$. Let $V_f = \{P \in V | f(P) \neq 0\}$, an open subset of V. Then show that
(a) $\Gamma(V_f) = \Gamma(V)[1/f] = \{a/f^n \in k(V) | a \in \Gamma(V), n \in \mathbb{Z}\}$.
(b) V_f is an affine variety.

Question 6
(a) Let $f : X \longrightarrow Y$ be a morphism of varieties such that $f(X)$ is dense in Y. Show that the homomorphism $\tilde{f} : \Gamma(Y) \longrightarrow \Gamma(X)$ is one to one.
(b) If X and Y are affine, show that $f(X)$ is dense in Y if and only if $\tilde{f} : \Gamma(Y) \longrightarrow \Gamma(X)$ is one to one.

Question 7
Let $k[x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^2$. Let $P = [0 : 0 : 1], P' = [0 : 1 : 0], P'' = [1 : 0 : 0]$ be the fundamental points in $\mathbb{P}^2$ and $Q : \mathbb{P}^2 \setminus \{P, P', P''\} \longrightarrow \mathbb{P}^2$ be given by the formula $Q([x : y : z]) = [yz : xz : xy]$. Let $F \in k[x, y, z]$ represents the curve C of degree n and $m_P(C) = r, m_{P'}(C) = r', m_{P''}(C) = r''$. Let $F^Q = F \circ Q = z^r y^{r'} x^{r''} F'$ and F' represents the curve C'.
(a) Show that $m_P(C') = n-r'-r'', m_{P'}(C') = n-r-r'', m_{P''}(C') = n-r-r'$.
(b) Show that if C is in good position (i.e, no exceptional lines are tangent to C at the fundamental points) then C' is also in good position.

Question 8
Let C be an irreducible projective curve, $f : X \longrightarrow C$ be the birational morphism from the nonsingular model X onto C and $K = k(C) = k(X)$ be the function field.
(a) For $z \in K$, define $div(z)$, divisor of z.
(b) Show that for any $z \in K$, $div(z)$ is a divisor of degree zero.
(c) Let us assume that the above curve C is a plane curve of degree n. Let G be a plane curve of degree m and do not contain C as a component. Define $div(G)$, divisor of G and show that $div(G)$ is a divisor of degree mn.